Degree of Convergence of an Integral Operator

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Abstract

In this paper we define an integral operator on L^p and obtain its degree of convergence in the appropriate norm. By specializing the kernel of the integral operator we obtain many known results as corollaries. We have also applied our results to obtain results on singular integral operators.

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1 Introduction

Let $L^p \equiv L^p(\mathbb{R})$ with fixed $1 \leq p \leq \infty$ be the space of all real-valued functions Lebesgue integrable to the $p-$th power over $\mathbb{R}$ if $1 \leq p < \infty$ and uniformly continuous and bounded on $\mathbb{R}$ if $p = \infty$. We define the norm in L^p, as usual, by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \begin{cases} \left\{ \int\limits_{\mathbb{R}} |f(x)|^p \, dx \right\}^{\frac{1}{p}} & \text{if} \quad 1 \leq p < \infty, \\ \sup\limits_{x \in \mathbb{R}} |f(x)| & \text{if} \quad p = \infty. \end{cases} \tag{1.1}$$

Denote by $\omega(f; \cdot)_p$ the modulus of continuity of $f \in L^p$, i.e.,

$$\omega(f; t)_p := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_p, \quad t \geq 0,$$

where $\Delta_h f(x) = f(x+h) - f(x)$.

Consider the family of integral operators

$$F_\lambda(f;x) := \lambda \int_{\mathbb{R}} f(t)\,\mathcal{K}(\lambda(t-x))\,dt, \quad \lambda > 0, \tag{1.2}$$

with Fejér type kernel $\mathcal{K}$ [1, p. 126]:

$$\mathcal{K}(-x) = \mathcal{K}(x), \tag{1.3}$$

$$\int_{\mathbb{R}} \mathcal{K}(x)\,dx = 1, \tag{1.4}$$

$$\sup_{-1\le x\le 1} |\mathcal{K}(x)| < \infty, \tag{1.5}$$

$$\sup_{x\in\mathbb{R}} x^2 |\mathcal{K}(x)| < \infty. \tag{1.6}$$

Under these conditions, the integral (1.2) represents a linear operator acting from L^p to L^p.

For fixed $m \in \mathbb{N}\cup\{0\}$ and $1 \le p \le \infty$, we denote by L^p_m the set of all $f \in L^p$ whose derivatives $f', f'', ..., f^{(m)}$ also belong to L^p. The norm in these L^p_m is defined by (1.1), i.e., for $f \in L^p_m$, we have $\|f\|_{p,\ m} = \|f\|_p$, where $\|f\|_p$ is defined by (1.1). It is clear that $L^p_0 \equiv L^p$.

Definition 1. Let $f \in L^p_m$ for fixed $m \in \mathbb{N}\cup\{0\}$ and $1 \le p \le \infty$. We define a family of modified integral operators by the formula

$$F_{\lambda,\ m}(f;x) := \lambda \int_{\mathbb{R}} \sum_{j=0}^{m} \frac{f^{(j)}(t)}{j!}(x-t)^j\,\mathcal{K}(\lambda(t-x))\,dt \tag{1.7}$$

for $x \in \mathbb{R}$ and $\lambda > 0$.

In particular, we have $F_{\lambda,\ 0}(f;\cdot) \equiv F_\lambda(f;\cdot)$ for $f \in L^p$.

It is obvious that the formula (1.7) can be rewritten in the following form:

$$F_{\lambda,\ m}(f;x) := \sum_{j=0}^{m} \frac{(-1)^j}{j!}\lambda \int_{\mathbb{R}} f^{(j)}(t+x)\,t^j\mathcal{K}(\lambda t)\,dt$$

for every $f \in L^p_m$, $x \in \mathbb{R}$ and $\lambda > 0$.

If (1.3) holds and for any $j = 0, 1, 2, ..., m$

$$\int_0^\infty u^j \left|\mathcal{K}(u)\right| < \infty,$$

then for fixed $m \in \mathbb{N} \cup \{0\}$ and $\lambda > 0$ the integral (1.7) is a linear operator from space L^p_m into L^p (see Remark 2).

Denote by $H^{\omega^*,\ p}$ the set of all functions $f \in L^p \left(1 \leq p \leq \infty\right)$ satisfying the condition

$$\sup_{h \neq 0} h^{\omega^*,\ p}\left(f; h\right) < \infty,$$

where

$$h^{\omega^*,\ p}\left(f; h\right) := \frac{\left\|\Delta_h f\left(\cdot\right)\right\|_p}{\omega^*\left(\left|h\right|\right)}, \qquad h^{0,\ p}\left(f; h\right) = 0$$

and, for $t \geq 0$, $\omega^*\left(t\right)$ is a nondecreasing function. We can show that $H^{\omega^*,\ p}$ is a Banach space with respect to the generalized Hölder norm

$$\left\|f\right\|_{\omega^*,\ p} := \left\|f\right\|_p + \sup_{h \neq 0} h^{\omega^*,\ p}\left(f; h\right). \tag{1.8}$$

Suppose that $H^{\omega,\ p}$ is the set of functions $f \in L^p \left(1 \leq p \leq \infty\right)$ satisfying the condition

$$\sup_{h \neq 0} h^{\omega,\ p}\left(f; h\right) = \sup_{h \neq 0} \frac{\left\|\Delta_h f\left(\cdot\right)\right\|_p}{\omega\left(\left|h\right|\right)} < \infty$$

and contained in the space $H^{\omega^*,\ p}$, $H^{\omega,\ p} \subset H^{\omega^*,\ p}$, where, for $t \geq 0$, $\omega\left(t\right)$ is a nondecreasing function. In particular, setting

$$\omega\left(t\right) = t^\alpha, \quad \omega^*\left(t\right) = t^\beta, \quad t \geq 0 \quad \text{and} \quad 0 \leq \beta < \alpha \leq 1\ ,$$

for $H^{\omega^*,\ p}$ we obtain the spaces

$$H^{\beta,\ p} := \left\{f \in L^p : \omega\left(f, t\right)_p \leq C_1\ t^\beta\right\}$$

with Hölder norm

$$\left\|f\right\|_{\beta,\ p} := \left\|f\right\|_p + \sup_{h \neq 0} \frac{\left\|\Delta_h f\left(\cdot\right)\right\|_p}{\left|h\right|^\beta},$$

and for the set $H^{\omega,\ p}$ we have

$$H^{\alpha,\ p} := \left\{f \in L^p : \omega\left(f, t\right)_p \leq C_2\ t^\alpha\right\},$$

$$H^{\alpha,\ p} \subset H^{\beta,\ p}.$$

For fixed $m \in \mathbb{N} \cup \{0\}$ and $1 \leq p \leq \infty$, we denote by $H_m^{\omega^*,\, p}$ (or $H_m^{\omega,\, p}$) the set of all $f \in H^{\omega^*,\, p}$ ($f \in H^{\omega,\, p}$) whose derivatives $f', f'', ..., f^{(m)}$ also belong to $H^{\omega^*,\, p}$ (or $H^{\omega,\, p}$), where, for $t \geq 0$, $\omega^*(t)$ (or $\omega(t)$) is a nondecreasing function. The norm in these $H_m^{\omega^*,\, p}$ is defined by (1.8), i.e., for $f \in H_m^{\omega^*,\, p}$, we have $\|f\|_{\omega^*,\, p,\, m} = \|f\|_{\omega^*,\, p}$, where $\|f\|_{\omega^*,\, p}$ is defined by (1.8). It is clear that $H_0^{\omega^*,\, p} \equiv H^{\omega^*,\, p}$.

Throughout the paper we shall use the following notation:

$$E_\lambda(x) = E_\lambda(f;x) = F_\lambda(f;x) - f(x),$$

$$E_\lambda(x+h, x) = E_\lambda(f; x+h, x) = E_\lambda(x+h) - E_\lambda(x)$$

and

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

The object of this paper is to study degree of convergence of the integral operator $F_{\lambda,\, m}(f)$ to f in the appropriate norm and to deduce many interesting results as corollaries. We also apply our results to obtain degree of convergence of singular integrals.

2 Statement of the results

Theorem 1. *Suppose that* $0 \leq \beta < \eta \leq 1$ *and (1.3)-(1.6) hold. Then, for any* $f \in H^{\omega,\, p}\,(1 \leq p \leq \infty)$, *the following relation is true*

$$\|F_\lambda(f) - f\|_{\omega^*,\, p} = O(1) \sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left\{ \frac{1}{\lambda} \int_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} du \right\}, \tag{2.1}$$

where $\lambda > 1$.

Remark 1. For $p = \infty$ (2.1) was proved in [6, Theorem 1].

Theorem 2. *Suppose that* $0 \leq \beta < \eta \leq 1$, $\lambda \geq \lambda_0 > 0$ *and (1.3)-(1.5) hold. If*

$$\int_1^\infty u\, |\mathcal{K}(u)|\, du < \infty, \tag{2.2}$$

then for $f \in H^{\omega,\, p}\,(1 \leq p \leq \infty)$

$$\|F_\lambda(f) - f\|_{\omega^*,\, p} = O(1) \sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Remark 2. Let $m\in\mathbb{N}\cup\{0\}$, $1\leq p\leq\infty$ and (1.3) holds. If for any $j=0,1,2,...,m$

$$\int\limits_{0}^{\infty}u^{j}\left|\mathcal{K}\left(u\right)\right|du<\infty, \tag{2.3}$$

then for every $f\in L_{m}^{p}$ and $\lambda>0$ we have

$$\left\|F_{\lambda,\ m}\left(f\right)\right\|_{p}=O\left(1\right)\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{p}}{j!\lambda^{j}}.$$

Moreover, if $f\in H_{m}^{\omega^{*},\ p}$ and $\lambda>0$, then

$$\left\|F_{\lambda,\ m}\left(f\right)\right\|_{\omega^{*},\ p}=O\left(1\right)\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{\omega^{*},\ p}}{j!\lambda^{j}}.$$

Theorem 3. *Suppose that $0\leq\beta<\eta\leq1$, $m\in\mathbb{N}$, $\lambda\geq\lambda_{0}>0$ and (1.3)-(1.4) hold. If for any $j=1,2,...,m+1$*

$$\int\limits_{0}^{\infty}u^{j}\left|\mathcal{K}\left(u\right)\right|du<\infty, \tag{2.4}$$

then for $f\in H_{m}^{\omega,\ p}\left(1\leq p\leq\infty\right)$

$$\left\|F_{\lambda,\ m}\left(f\right)-f\right\|_{\omega^{*},\ p}=O\left(1\right)\sup_{h\neq0}\frac{\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}}{\omega^{*}\left(\left|h\right|\right)}\frac{1}{\lambda^{m}}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Setting

$$\omega\left(t\right)=t^{\alpha},\quad\omega^{*}\left(t\right)=t^{\beta},\quad0\leq\beta<\alpha\leq1,\quad\eta=\alpha$$

in the assumptions of Theorem 1, 2 and 3, we obtain the following corollaries:

Corollary 1. *Suppose that $0\leq\beta<\alpha\leq1$ and (1.3)-(1.6) hold. Then, for any $f\in H^{\alpha,\ p}\left(1\leq p\leq\infty\right)$, the following relation is true*

$$\left\|F_{\lambda}\left(f\right)-f\right\|_{\beta,\ p}=\begin{cases}O\left(\lambda^{\beta-\alpha}\right) & \text{if}\quad\alpha-\beta<1,\\ O\left(\frac{\ln\lambda}{\lambda}\right) & \text{if}\quad\alpha-\beta=1,\end{cases}$$

where $\lambda>1$.

Corollary 2. *Suppose that $0\leq\beta<\alpha\leq1$, $\lambda\geq\lambda_{0}>0$ and (1.3)-(1.5) hold. If*

$$\int\limits_{1}^{\infty}u\left|\mathcal{K}\left(u\right)\right|du<\infty,$$

then for $f\in H^{\alpha,\ p}\left(1\leq p\leq\infty\right)$

$$\left\|F_{\lambda}\left(f\right)-f\right\|_{\beta,\ p}=O\left(\lambda^{\beta-\alpha}\right).$$

Corollary 3. *Suppose that* $0 \leq \beta < \alpha \leq 1$, $m \in \mathbb{N}$, $\lambda \geq \lambda_0 > 0$ *and (1.3)-(1.4) hold. If for any* $j = 1, 2, ..., m+1$

$$\int_0^\infty u^j \left| \mathcal{K}(u) \right| du < \infty,$$

then for $f \in H_m^{\alpha,\ p}$ $(1 \leq p \leq \infty)$

$$\left\| F_{\lambda,\ m}(f) - f \right\|_{\beta,\ p} = O\left(\lambda^{\beta-\alpha-m}\right).$$

3 Examples

3.1 The Riesz means of the Fourier series

Suppose that the kernel $\mathcal{K}$ satisfies the conditions (1.3)-(1.6). Then, as is well known (see [1, p. 132]), if we consider 2π periodic function $f \in L^p$ with the Fourier series

$$S(f) = \sum_{k=-\infty}^{\infty} c_k(f) e^{ikx},$$

the family of operators of Fejér type (1.2) can be transformed into sequences of linear means of series of the function f

$$F_n(f; x) = \sum_{k=-\infty}^{\infty} \varphi\left(\frac{k}{n}\right) c_k(f) e^{ikx}, \quad n \in \mathbb{N},$$

where the values of $\varphi\left(\frac{k}{n}\right)$ coincide for $t = \frac{k}{n}$ with the values of the function $\varphi(t)$, which is the Fourier transform of the kernel $\mathcal{K}$.

Consider the Riesz means of the Fourier series $S(f)$:

$$R_n(\gamma, f; x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right)^\gamma c_k(f) e^{ikx}, \quad \gamma > 0.$$

For this mean

$$\varphi_\gamma(t) = \begin{cases} (1-|t|)^\gamma, & |t| \leq 1, \\ 0, & |t| \geq 1, \end{cases}$$

and consequently

$$\mathcal{K}_{R(\gamma)}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_\gamma(x) e^{itx} dx$$

$$= \frac{1}{\pi}\left\{\cos t \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!\,(2k+1+\gamma)} + \sin t \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!\,(2k+2+\gamma)}\right\}. \tag{3.1}$$

It is clear that the kernel $\mathcal{K}_{R(\gamma)}$ satisfies the conditions (1.3), (1.5) and (1.6). Moreover, the function φ_γ is the Fourier transform of the kernel $\mathcal{K}_{R(\gamma)}$, i.e.,

$$\varphi_\gamma(t) = \int_{\mathbb{R}} \mathcal{K}_{R(\gamma)}(x)\, e^{-itx} dx$$

for all $t \in \mathbb{R}$. Thus

$$1 = \varphi_\gamma(0) = \int_{\mathbb{R}} \mathcal{K}_{R(\gamma)}(x)\, dx$$

and the condition (1.4) is valid, too. Hence, by Theorem 1, we obtain the following results.

Corollary 4. *Suppose that $0 \le \beta < \eta \le 1$ and $\gamma > 0$. Then, for any 2π periodic function $f \in H^{\omega,\, p}\,(1 \le p \le \infty)$, the following relation is true:*

$$\|R_n(\gamma, f) - f\|_{\omega^*,\ p} = O(1) \sup_{h \ne 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left\{\frac{1}{n}\sum_{k=1}^{n}\left(\omega\left(\frac{1}{k}\right)\right)^{1-\frac{\beta}{\eta}}\right\}.$$

Corollary 5. *Suppose that $0 \le \beta < \alpha \le 1$ and $\gamma > 0$. Then, for any 2π periodic function $f \in H^{\alpha,\, p}\,(1 \le p \le \infty)$, the following relation holds:*

$$\|R_n(\gamma, f) - f\|_{\beta,\ p} = \begin{cases} O\left(n^{\beta-\alpha}\right) & \text{if} \quad \alpha - \beta < 1, \\ O\left(\frac{\ln n}{n}\right) & \text{if} \quad \alpha - \beta = 1. \end{cases}$$

In particular, putting $\gamma = 1$ in (3.1), we get the Fejér kernel

$$\mathcal{K}_\sigma(t) := \mathcal{K}_{R(1)}(t) = \frac{2}{\pi}\left(\frac{\sin(t/2)}{t}\right)^2$$

and consequently the Fejér means of the Fourier series $S(f)$ is given by

$$\sigma_n(f; x) := R_n(1, f; x) = \sum_{k=-\infty}^{\infty} \varphi\left(\frac{\kappa}{n}\right) c_k(f)\, e^{ikx} = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n}\right) c_k(f)\, e^{ikx}.$$

Hence, from the above corollaries, we obtain the results from [3], [6] and [10].

3.2 The Poisson operator

Let

$$\mathcal{K}_{\overline{P}}(t) = \frac{1}{\pi}\frac{1}{1+t^2}, \quad \lambda = \frac{1}{\varepsilon}, \quad \varepsilon > 0.$$

Then we obtain from (1.2) the Poisson singular integral of a function $f \in L^p$, i.e

$$\overline{P}_\varepsilon (f; x) = \frac{\varepsilon}{\pi} \int\limits_{\mathbb{R}} f(x+t) \frac{1}{\varepsilon^2 + t^2} dt,$$

The approximation properties of this integral was given in [5], for example. It is clear that the kernel $\mathcal{K}_{\overline{P}}$ satisfies conditions (1.3)-(1.6). Hence, by Theorem 1, we obtain the following assertion.

Corollary 6. *Suppose that $0 \leq \beta < \eta \leq 1$. Then, for any 2π periodic function $f \in H^{\omega,\ p}\,(1 \leq p \leq \infty)$, the following relation is true*

$$\left\| \overline{P}_\varepsilon (f) - f \right\|_{\omega^*,\ p} = O(1) \sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left\{ \varepsilon \int\limits_1^{1/\varepsilon} \left(\omega \left(\frac{1}{u} \right) \right)^{1-\frac{\beta}{\eta}} du \right\},$$

where $0 < \varepsilon < 1$.

Corollary 7. *Suppose that $0 \leq \beta < \alpha \leq 1$. Then, for any 2π periodic function $f \in H^{\alpha,\ p}\,(1 \leq p \leq \infty)$, the following relation is true*

$$\left\| \overline{P}_\varepsilon (f) - f \right\|_{\beta,\ p} = \begin{cases} O\left(\varepsilon^{\beta-\alpha}\right) & \text{if} \quad \alpha - \beta < 1, \\ O\left(\varepsilon \ln(1/\varepsilon)\right) & \text{if} \quad \alpha - \beta = 1, \end{cases}$$

as $\varepsilon \to 0^+$.

3.3 The Picard and Gauss-Weierstrass operators

Taking

$$\mathcal{K}_P(t) = \frac{1}{2} \exp(-|t|), \quad \lambda = \frac{1}{r}, \quad r > 0$$

and

$$\mathcal{K}_W(t) = \frac{1}{\sqrt{\pi}} \exp\left(-t^2\right), \quad \lambda = \frac{1}{\sqrt{2r}}, \quad r > 0$$

we obtain from (1.2), respectively, the Picard singular integral and the Gauss-Weierstrass singular integrals of a function $f \in L^p$, i.e.,

$$P_r(f; x) = \frac{1}{2r} \int\limits_{\mathbb{R}} f(x+t) \exp\left(\frac{-|t|}{r} \right) dt,$$

$$W_r(f; x) = \frac{1}{\sqrt{4\pi r}} \int\limits_{\mathbb{R}} f(x+t) \exp\left(-\frac{t^2}{4r} \right) dt.$$

The limit properties (as $r \to 0^+$) of these integrals were given in many papers and monographs (see, e.g., [2, 3, 4]).

It is clear (see [6, Lemma 1]) that for every $m \in \mathbb{N} \cup \{0\}$

$$\int_0^\infty u^m \left|\mathcal{K}_P(u)\right| du = \frac{m!}{2}$$

and

$$\int_0^\infty u^m \left|\mathcal{K}_W(u)\right| du = \begin{cases} \frac{1}{2} & \text{if} \quad m = 0, \\ \frac{(2k-1)!!}{2^{k+1}} & \text{if} \quad m = 2k \geq 2, \\ \frac{k!}{2\sqrt{\pi}} & \text{if} \quad m = 2k+1 \geq 1. \end{cases}$$

Hence, by Theorems 2 and 3, we obtain the following assertion.

Corollary 8. *Suppose that* $0 \leq \beta < \eta \leq 1$, $m \in \mathbb{N} \cup \{0\}$ *and* $f \in H_m^{\omega,\, p}$ $(1 \leq p \leq \infty)$. *Then*

$$\|P_r(f) - f\|_{\omega^*,\, p} = O(1) \sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} r^m (\omega(r))^{1-\frac{\beta}{\eta}}$$

and

$$\|W_r(f) - f\|_{\omega^*,\, p} = O(1) \sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} r^{m/2} \left(\omega\left(\sqrt{r}\right)\right)^{1-\frac{\beta}{\eta}}$$

as $r \to 0^+$.

Corollary 9. *Suppose that* $0 \leq \beta < \alpha \leq 1$, $m \in \mathbb{N} \cup \{0\}$ *and* $f \in H_m^{\alpha,\, p}$ $(1 \leq p \leq \infty)$. *Then*

$$\|P_r(f) - f\|_{\beta,\, p} = O\left(r^{m+\alpha-\beta}\right)$$

and

$$\|W_r(f) - f\|_{\beta,\, p} = O\left(r^{(m+\alpha-\beta)/2}\right)$$

as $r \to 0^+$.

4 Proofs of the theorems

4.1 Proof of Theorem 1

For $p = \infty$, (2.1) was proved in [6, Theorem 1]. Let $1 \leq p < \infty$. Then using (1.3) and (1.4) we get

$$E_\lambda(x) = \lambda \int_0^\infty \phi_x(t) \mathcal{K}(\lambda t)\, dt$$

and

$$E_\lambda (x+h,x)=\lambda \int\limits_0^\infty (\phi_{x+h}(t)-\phi_x(t))\,\mathcal{K}(\lambda t)\,dt.$$

Applying the Fubini inequality [11] we have

$$\|E_\lambda(\cdot+h,\cdot)\|_p=\lambda\left\{\int\limits_{\mathbb{R}}\left|\int\limits_0^\infty (\phi_{x+h}(t)-\phi_x(t))\,\mathcal{K}(\lambda t)\,dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\leq \lambda\int\limits_0^\infty |\mathcal{K}(\lambda t)|\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_x(t)|^p\,dx\right\}^{\frac{1}{p}} dt$$

$$=\lambda\left(\int\limits_0^{1/\lambda}+\int\limits_{1/\lambda}^{1}+\int\limits_1^\infty\right)|\mathcal{K}(\lambda t)|\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_x(t)|^p\,dx\right\}^{\frac{1}{p}} dt=I_1+I_1+I_3. \quad (4.1)$$

It is clear that for $1\leq p<\infty$,

$$\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_x(t)|^p\,dx\right\}^{\frac{1}{p}}\leq 4\omega(f,|h|)_p \quad (4.2)$$

and $t\geq 0$

$$\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_x(t)|^p\,dx\right\}^{\frac{1}{p}}\leq 4\omega(f,t)_p\,. \quad (4.3)$$

Then, in view of the property (1.5) of the kernel $\mathcal{K}$ and inequalities (4.2) and (4.3), we obtain that for $f\in H^{\omega,\,p}$

$$I_1=\lambda\int\limits_0^{1/\lambda}|\mathcal{K}(\lambda t)|\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_x(t)|^p\,dx\right\}^{\frac{1}{p}\frac{\beta}{\eta}}$$

$$\cdot\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_x(t)|^p\,dx\right\}^{\frac{1}{p}\left(1-\frac{\beta}{\eta}\right)} dt$$

$$=O(1)\,\lambda\,(\omega(|h|))^{\frac{\beta}{\eta}}\int\limits_0^{1/\lambda}|\mathcal{K}(\lambda t)|\left(\omega(f,t)_p\right)^{1-\frac{\beta}{\eta}} dt$$

$$=O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}. \quad (4.4)$$

Further, in view of property (1.6) of the kernel $\mathcal{K}$, by (4.2) and (4.3) we have that for $f \in H^{\omega,\ p}$

$$I_2 = O(1)\,\lambda\,(\omega(|h|))^{\frac{\beta}{\eta}} \int\limits_{1/\lambda}^{1} |\mathcal{K}(\lambda t)| \left(\omega(f,t)_p\right)^{1-\frac{\beta}{\eta}} dt$$

$$= O(1)\,\lambda\,(\omega(|h|))^{\frac{\beta}{\eta}} \int\limits_{1/\lambda}^{1} \left(\omega(f,t)_p\right)^{1-\frac{\beta}{\eta}} \frac{1}{(\lambda t)^2} dt$$

$$= O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \frac{1}{\lambda} \int\limits_{1}^{\lambda} \left(\omega\left(f;\frac{1}{u}\right)_p\right)^{1-\frac{\beta}{\eta}} du$$

$$= O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \frac{1}{\lambda} \int\limits_{1}^{\lambda} \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} du. \tag{4.5}$$

Applying (1.6), (4.2) we get for $f \in H^{\omega,\ p}$ that

$$I_3 = \lambda \int\limits_{0}^{1/\lambda} |\mathcal{K}(\lambda t)| \left\{ \int\limits_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p\, dx \right\}^{\frac{1}{p}\frac{\beta}{\eta}}$$

$$\cdot \left\{ \int\limits_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p\, dx \right\}^{\frac{1}{p}\left(1-\frac{\beta}{\eta}\right)} dt$$

$$= O(1)\,\lambda\,(\omega(|h|))^{\frac{\beta}{\eta}} \int\limits_{1}^{\infty} |\mathcal{K}(\lambda t)| \left(8\,\|f\|_p\right)^{1-\frac{\beta}{\eta}} dt$$

$$= O(1)\,\lambda\,(\omega(|h|))^{\frac{\beta}{\eta}} \int\limits_{1}^{\infty} \frac{1}{(\lambda t)^2} dt = O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \frac{1}{\lambda}$$

$$= O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \frac{1}{\lambda^{\frac{\beta}{\eta}}} \frac{1}{\lambda^{1-\frac{\beta}{\eta}}}.$$

If $\lambda > 1$ then $\omega(f;1)_p \leq 2\lambda\omega\left(f,\frac{1}{\lambda}\right)$. Thus

$$I_3 = O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(f,\frac{1}{\lambda}\right)_p\right)^{1-\frac{\beta}{\eta}} \frac{1}{\lambda^{\frac{\beta}{\eta}}}$$

$$= O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}. \tag{4.6}$$

From relation (4.1), (4.4), (4.5) and (4.6), we obtain

$$\left\| E_\lambda\left(\cdot+h,\cdot\right)\right\|_p = O\left(1\right)\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}\left\{\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}} + \frac{1}{\lambda}\int\limits_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} du\right\}.$$

Since

$$\frac{1}{\lambda}\int\limits_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} du \geq \frac{1}{\lambda}\int\limits_{\frac{\lambda-1}{2}}^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} du$$

$$\geq \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}} \frac{\lambda+1}{2\lambda} \geq \frac{1}{2}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}},$$

therefore

$$\left\| E_\lambda\left(\cdot+h,\cdot\right)\right\|_p = O\left(1\right)\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}\left\{\frac{1}{\lambda}\int\limits_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} du\right\}.$$

Hence

$$\sup_{h\neq 0}\frac{\left\| E_\lambda\left(\cdot+h,\cdot\right)\right\|_p}{\omega^*\left(\left|h\right|\right)} = O\left(1\right)\frac{\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}}{\omega^*\left(\left|h\right|\right)}\left\{\frac{1}{\lambda}\int\limits_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} du\right\}. \tag{4.7}$$

We can easily see that

$$\left\| E_\lambda\left(\cdot\right)\right\|_p = O\left(1\right)\frac{1}{\lambda}\int\limits_1^\lambda \omega\left(\frac{1}{u}\right) du = O\left(1\right)\frac{1}{\lambda}\int\limits_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} du. \tag{4.8}$$

From (4.7) an (4.8), we finally obtain

$$\left\| E_\lambda\left(\cdot\right)\right\|_{\omega^*,\ p} = \left\| E_\lambda\left(\cdot\right)\right\|_p + \sup_{h\neq 0}\frac{\left\| E_\lambda\left(\cdot+h,\cdot\right)\right\|_p}{\omega^*\left(\left|h\right|\right)}$$

$$= O\left(1\right)\frac{\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}}{\omega^*\left(\left|h\right|\right)}\left\{\frac{1}{\lambda}\int\limits_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} du\right\}.$$

This completes the proof of Theorem 1. □

4.2 Proof of Theorem 2

Let $p=\infty$. Then by (1.3) and (1.4) we get

$$\left|E_\lambda\left(x+h,x\right)\right| \leq \lambda\int\limits_0^\infty \left|\mathcal{K}\left(\lambda t\right)\right|\left|\phi_{x+h}\left(t\right)-\phi_x\left(t\right)\right| dt$$

$$= \lambda \left(\int_0^{1/\lambda} + \int_{1/\lambda}^{\infty} \right) |\mathcal{K}(\lambda t)| \left| \phi_{x+h}(t) - \phi_x(t) \right| dt = J_1 + J_2. \tag{4.9}$$

It is clear that

$$\left| \phi_{x+h}(t) - \phi_x(t) \right| \leq 4\omega (f; |h|)_\infty$$

and

$$\left| \phi_{x+h}(t) - \phi_x(t) \right| \leq 4\omega (f; t)_\infty . \tag{4.10}$$

Using this and (1.5) we have that for $f \in H^{\omega,\ p}$

$$J_1 = \lambda \int_0^{1/\lambda} |\mathcal{K}(\lambda t)| \left(\left| \phi_{x+h}(t) - \phi_x(t) \right| \right)^{\frac{\beta}{\eta}} \left(\left| \phi_{x+h}(t) - \phi_x(t) \right| \right)^{1-\frac{\beta}{\eta}} dt$$

$$= O(1) \lambda \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \int_0^{1/\lambda} \left(\omega(f;t)_\infty \right)^{1-\frac{\beta}{\eta}} dt = O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right) \right)^{1-\frac{\beta}{\eta}} . \tag{4.11}$$

Further, by (4.10) and (2.2) we obtain that for $f \in H^{\omega,\ p}$

$$J_2 = \lambda \int_{1/\lambda}^{\infty} |\mathcal{K}(\lambda t)| \left(\left| \phi_{x+h}(t) - \phi_x(t) \right| \right)^{\frac{\beta}{\eta}} \left(\left| \phi_{x+h}(t) - \phi_x(t) \right| \right)^{1-\frac{\beta}{\eta}} dt$$

$$= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \lambda \int_{1/\lambda}^{\infty} \left(\omega(f;t)_\infty \right)^{1-\frac{\beta}{\eta}} |\mathcal{K}(\lambda t)| \, dt$$

$$= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \lambda \int_{1/\lambda}^{\infty} \left(\frac{\omega(f;t)_\infty}{t} \right)^{1-\frac{\beta}{\eta}} t^{1-\frac{\beta}{\eta}} |\mathcal{K}(\lambda t)| \, dt$$

$$= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \left(\omega\left(f;\frac{1}{\lambda}\right)_\infty \right)^{1-\frac{\beta}{\eta}} \lambda^{2-\frac{\beta}{\eta}} \int_{1/\lambda}^{\infty} t^{1-\frac{\beta}{\eta}} |\mathcal{K}(\lambda t)| \, dt$$

$$= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right) \right)^{1-\frac{\beta}{\eta}} \lambda^2 \int_{1/\lambda}^{\infty} t \, |\mathcal{K}(\lambda t)| \, dt$$

$$= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right) \right)^{1-\frac{\beta}{\eta}} \int_1^{\infty} u \, |\mathcal{K}(u)| \, du$$

$$= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right) \right)^{1-\frac{\beta}{\eta}} . \tag{4.12}$$

Similarly we can prove that

$$\|E_{\lambda}(\cdot)\|_{\infty}=O(1)\,\omega\left(f;\frac{1}{\lambda}\right)_{\infty}=O(1)\left(\omega\left(f;\frac{1}{\lambda}\right)_{\infty}\right)^{1-\frac{\beta}{\eta}}\left(\omega\left(f;\frac{1}{\lambda_0}\right)_{\infty}\right)^{\frac{\beta}{\eta}}$$

$$=O(1)\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}. \tag{4.13}$$

Hence, by (4.9), (4.11), (4.12) and (4.13)

$$\|E_{\lambda}(\cdot)\|_{\omega^{*},\,\infty}=\|E_{\lambda}(\cdot)\|_{\infty}+\sup_{h\neq 0}\frac{\|E_{\lambda}(\cdot+h,\cdot)\|_{\infty}}{\omega^{*}(|h|)}$$

$$=O(1)\sup_{h\neq 0}\frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^{*}(|h|)}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Suppose $1\leq p<\infty$. Then using (1.3), (1.4) and the Fubini inequality [11] we get

$$\|E_{\lambda}(\cdot+h,\cdot)\|_{p}=\lambda\left\{\int\limits_{\mathbb{R}}\left|\int\limits_{0}^{\infty}(\phi_{x+h}(t)-\phi_{x}(t))\,\mathcal{K}(\lambda t)\,dt\right|^{p}dx\right\}^{\frac{1}{p}}$$

$$\leq\lambda\int\limits_{0}^{\infty}|\mathcal{K}(\lambda t)|\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_{x}(t)|^{p}\,dx\right\}^{\frac{1}{p}}dt$$

$$=\lambda\left(\int\limits_{0}^{1/\lambda}+\int\limits_{1/\lambda}^{\infty}\right)|\mathcal{K}(\lambda t)|\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_{x}(t)|^{p}\,dx\right\}^{\frac{1}{p}}dt=S_1+S_2. \tag{4.14}$$

In view of property (1.5) of the kernel $\mathcal{K}$ and inequalities (4.2) and (4.3), we obtain that for $f\in H^{\omega,\,p}$

$$S_1=\lambda\int\limits_{0}^{1/\lambda}|\mathcal{K}(\lambda t)|\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_{x}(t)|^{p}\,dx\right\}^{\frac{1}{p}\frac{\beta}{\eta}}$$

$$\cdot\left\{\int\limits_{\mathbb{R}}|\phi_{x+h}(t)-\phi_{x}(t)|^{p}\,dx\right\}^{\frac{1}{p}\left(1-\frac{\beta}{\eta}\right)}dt$$

$$=O(1)\,\lambda\,(\omega(|h|))^{\frac{\beta}{\eta}}\int\limits_{0}^{1/\lambda}|\mathcal{K}(\lambda t)|\left(\omega(f,t)_{p}\right)^{1-\frac{\beta}{\eta}}dt$$

$$=O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}. \tag{4.15}$$

Further, in view of property (2.2) of the kernel $\mathcal{K}$, by (4.2) and (4.3) we have for $f \in H^{\omega,\ p}$

$$S_2 = O(1)\,\lambda\,(\omega(|h|))^{\frac{\beta}{\eta}} \int\limits_{1/\lambda}^{\infty} |\mathcal{K}(\lambda t)| \left(\omega(f,t)_p\right)^{1-\frac{\beta}{\eta}} dt$$

$$= O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}}\,\lambda \int\limits_{1/\lambda}^{\infty} \left(\frac{\omega(f;t)_p}{t}\right)^{1-\frac{\beta}{\eta}} t^{1-\frac{\beta}{\eta}}\,|\mathcal{K}(\lambda t)|\,dt$$

$$= O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(f;\frac{1}{\lambda}\right)_p\right)^{1-\frac{\beta}{\eta}} \lambda^{2-\frac{\beta}{\eta}} \int\limits_{1/\lambda}^{\infty} t^{1-\frac{\beta}{\eta}}\,|\mathcal{K}(\lambda t)|\,dt$$

$$= O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}} \lambda^{2} \int\limits_{1/\lambda}^{\infty} t\,|\mathcal{K}(\lambda t)|\,dt$$

$$= O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}} \int\limits_{1}^{\infty} u\,|\mathcal{K}(u)|\,du$$

$$= O(1)\,(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}. \tag{4.16}$$

We can easily see that

$$\|E_\lambda(\cdot)\|_p = O(1)\,\omega\left(f;\frac{1}{\lambda}\right)_p = O(1)\left(\omega\left(f;\frac{1}{\lambda_0}\right)\right)^{\frac{\beta}{\eta}} \left(\omega\left(f;\frac{1}{\lambda}\right)_p\right)^{1-\frac{\beta}{\eta}}$$

$$= O(1)\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}. \tag{4.17}$$

Hence, by (4.14) -(4.17)

$$\|E_\lambda(\cdot)\|_{\omega^*,\ p} = \|E_\lambda(\cdot)\|_p + \sup_{h\neq 0} \frac{\|E_\lambda(\cdot+h,\cdot)\|_p}{\omega^*(|h|)}$$

$$= O(1) \sup_{h\neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

The proof is complete. $\square$

4.2.1 Proof of Remark 2

Let $p=\infty$. Then by (1.3) and (2.3) we get

$$\left\|F_{\lambda,\ m}(f)\right\|_{\infty}\leq\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\lambda\int\limits_{\mathbb{R}}\left|t^{j}\mathcal{K}(\lambda t)\right|dt=2\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\lambda\int\limits_{0}^{\infty}t^{j}\left|\mathcal{K}(\lambda t)\right|dt$$

$$=2\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!\lambda^{j}}\int\limits_{0}^{\infty}u^{j}\left|\mathcal{K}(u)\right|du=O(1)\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!\lambda^{j}}. \tag{4.18}$$

Suppose that $1\leq p<\infty$. Then using the Fubini inequality [11], (1.3) and (2.3) we obtain

$$\left\|F_{\lambda,\ m}(f)\right\|_{p}=\left\|\sum_{j=0}^{m}\frac{(-1)^{j}}{j!}\lambda\int\limits_{\mathbb{R}}f^{(j)}(t+\cdot)\,t^{j}\mathcal{K}(\lambda t)\,dt\right\|_{p}$$

$$\leq\sum_{j=0}^{m}\frac{\lambda}{j!}\left\{\int\limits_{\mathbb{R}}\left|\int\limits_{\mathbb{R}}f^{(j)}(t+x)\,t^{j}\mathcal{K}(\lambda t)\,dt\right|^{p}dx\right\}^{\frac{1}{p}}$$

$$\leq\sum_{j=0}^{m}\frac{\lambda}{j!}\int\limits_{\mathbb{R}}\left|t^{j}\mathcal{K}(\lambda t)\right|\left\{\int\limits_{\mathbb{R}}\left|f^{(j)}(t+x)\right|^{p}dx\right\}^{\frac{1}{p}}$$

$$=\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{p}}{j!}\lambda\int\limits_{0}^{\infty}t^{j}\left|\mathcal{K}(\lambda t)\right|dt=O(1)\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{p}}{j!\lambda^{j}}. \tag{4.19}$$

Using (4.18) and (4.19) we get

$$\left\|F_{\lambda,\ m}(f)\right\|_{\omega^{*},\ p}=\left\|F_{\lambda,\ m}(f)\right\|_{p}+\sup_{h\neq0}\frac{\left\|\Delta_{h}F_{\lambda,\ m}(f;\cdot)\right\|_{p}}{\omega^{*}(|h|)}$$

$$=\left\|F_{\lambda,\ m}(f)\right\|_{p}+\sup_{h\neq0}\frac{\left\|F_{\lambda,\ m}(\Delta_{h}f;\cdot)\right\|_{p}}{\omega^{*}(|h|)}$$

$$=O(1)\left\{\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{p}}{j!\lambda^{j}}+\sup_{h\neq0}\sum_{j=0}^{m}\frac{\left\|\Delta_{h}f^{(j)}(\cdot)\right\|_{p}}{j!\lambda^{j}\omega^{*}(|h|)}\right\}=O(1)\sum_{j=0}^{m}\frac{\left\|f^{(j)}\right\|_{\omega^{*},p}}{j!\lambda^{j}}.$$

This ends our proof. $\square$

4.3 Proof of Theorem 3

We use the following modified Taylor formula for $f \in L_m^p$ with $m \in \mathbb{N}$:

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(t)}{j!} (x-t)^j$$

$$+\frac{(x-t)^m}{(m-1)!} \int_0^1 (1-u)^{m-1} \left\{ f^{(m)}(t+u(x-t)) - f^{(m)}(t) \right\} du$$

for a fixed $t \in \mathbb{R}$ and every $x \in \mathbb{R}$.

By (1.3) we get

$$f(x) = \lambda \int_{\mathbb{R}} f(x) \mathcal{K}(\lambda(t-x))\, dt = \lambda \int_{\mathbb{R}} \sum_{j=0}^{m} \frac{f^{(j)}(t)}{j!} (x-t)^j \mathcal{K}(\lambda(t-x))\, dt$$

$$+\lambda \int_{\mathbb{R}} \mathcal{K}(\lambda(t-x)) \frac{(x-t)^m}{(m-1)!}$$

$$\cdot \left(\int_0^1 (1-u)^{m-1} \left\{ f^{(m)}(t+u(x-t)) - f^{(m)}(t) \right\} du \right) dt$$

$$= F_{\lambda,m}(f;x) + \lambda \int_{\mathbb{R}} \mathcal{K}(\lambda(t-x)) \frac{(x-t)^m}{(m-1)!}$$

$$\cdot \left(\int_0^1 (1-u)^{m-1} \left\{ f^{(m)}(t+u(x-t)) - f^{(m)}(t) \right\} du \right) dt.$$

Therefore, by (1.3)

$$f(x) - F_{\lambda,\, m}(f;x)$$

$$= \lambda \int_{\mathbb{R}} \left(\frac{(x-t)^m}{(m-1)!} \int_0^1 (1-u)^{m-1} \Delta_{u(x-t)} f^{(m)}(t)\, du \right) \mathcal{K}(\lambda(t-x))\, dt$$

$$= \lambda \int_{\mathbb{R}} \left(\frac{t^m}{(m-1)!} \int_0^1 (1-u)^{m-1} \Delta_{ut} f^{(m)}(x-t)\, du \right) \mathcal{K}(\lambda t)\, dt. \tag{4.20}$$

Set

$$E_{\lambda,\, m}(x) = E_{\lambda,\, m}(f;x) := f(x) - F_{\lambda,\, m}(f;x)$$

and

$$E_{\lambda,\, m}(x+h,x) = E_{\lambda,\, m}(f;x) := E_{\lambda,\, m}(x+h) - E_{\lambda,\, m}(x).$$

Let $p=\infty$. Then

$$\left|E_{\lambda,\ m}\left(x+h,x\right)\right|\leq\lambda\int\limits_{\mathbb{R}}\left|\mathcal{K}\left(\lambda t\right)\right|\frac{\left|t\right|^{m}}{\left(m-1\right)!}$$

$$\cdot\left(\int\limits_{0}^{1}\left(1-u\right)^{m-1}\left|\Delta_{ut}f^{(m)}\left(x+h-t\right)-\Delta_{ut}f^{(m)}\left(x-t\right)\right|du\right)dt.$$

It is clear that

$$\left|\Delta_{ut}f^{(m)}\left(x+h-t\right)-\Delta_{ut}f^{(m)}\left(x-t\right)\right|\leq 2\omega\left(f^{(m)};\left|ut\right|\right)_{\infty}$$

and

$$\left|\Delta_{ut}f^{(m)}\left(x+h-t\right)-\Delta_{ut}f^{(m)}\left(x-t\right)\right|\leq 2\omega\left(f^{(m)};\left|h\right|\right)_{\infty}.$$

This and the properties of the modulus of continuity yields for $f\in H_{m}^{\omega,\ p}$

$$\left|E_{\lambda,\ m}\left(x+h,x\right)\right|\leq\lambda\left(\omega\left(f^{(m)};\left|h\right|\right)_{\infty}\right)^{\frac{\beta}{\eta}}\int\limits_{\mathbb{R}}\frac{\left|t\right|^{m}}{\left(m-1\right)!}\left|\mathcal{K}\left(\lambda t\right)\right|$$

$$\cdot\left(\int\limits_{0}^{1}\left(1-u\right)^{m-1}\left(\omega\left(f^{(m)};\left|ut\right|\right)_{\infty}\right)^{1-\frac{\beta}{\eta}}du\right)dt$$

$$=O\left(1\right)\lambda\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}$$

$$\cdot\int\limits_{\mathbb{R}}\left(\frac{\left|t\right|^{m}}{\left(m-1\right)!}\left|\mathcal{K}\left(\lambda t\right)\right|\left(\omega\left(f^{(m)};\left|t\right|\right)_{\infty}\right)^{1-\frac{\beta}{\eta}}\int\limits_{0}^{1}\left(1-u\right)^{m-1}du\right)dt$$

$$=O\left(1\right)\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}\left(\omega\left(f^{(m)};\frac{1}{\lambda}\right)_{\infty}\right)^{1-\frac{\beta}{\eta}}\lambda\int\limits_{\mathbb{R}}\frac{\left|t\right|^{m}}{m!}\left|\mathcal{K}\left(\lambda t\right)\right|\left(1+\lambda\left|t\right|\right)^{1-\frac{\beta}{\eta}}dt.$$

Using (1.3) and (2.4)

$$=O\left(1\right)\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}\left(\omega\left(f^{(m)};\frac{1}{\lambda}\right)_{\infty}\right)^{1-\frac{\beta}{\eta}}\frac{\lambda}{m!}\int\limits_{0}^{\infty}t^{m}\left(1+\lambda t\right)\left|\mathcal{K}\left(\lambda t\right)\right|dt$$

$$=O\left(1\right)\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}\left(\lambda\int\limits_{0}^{\infty}t^{m}\left|\mathcal{K}\left(\lambda t\right)\right|dt+\lambda^{2}\int\limits_{0}^{\infty}t^{m+1}\left|\mathcal{K}\left(\lambda t\right)\right|dt\right)$$

$$=O\left(1\right)\left(\omega\left(\left|h\right|\right)\right)^{\frac{\beta}{\eta}}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}\frac{1}{\lambda^{m}}\left(\int\limits_{0}^{\infty}u^{m}\left|\mathcal{K}\left(u\right)\right|du+\int\limits_{0}^{\infty}u^{m+1}\left|\mathcal{K}\left(u\right)\right|du\right)$$

$$= O(1)\left(\omega(|h|)\right)^{\frac{\beta}{\eta}}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}\frac{1}{\lambda^m}. \tag{4.21}$$

We can easily see that

$$\|E_{\lambda,\ m}(\cdot)\|_{\infty} = O(1)\frac{1}{\lambda^m}\omega\left(f^{(m)};\frac{1}{\lambda}\right)_{\infty} = O(1)\frac{1}{\lambda^m}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}. \tag{4.22}$$

Hence, by (4.21) and (4.22)

$$\|E_{\lambda,\ m}(\cdot)\|_{\omega^*,\ \infty} = \|E_{\lambda,\ m}(\cdot)\|_{\infty} + \sup_{h\neq 0}\frac{\|E_{\lambda,\ m}(\cdot+h,\cdot)\|_{\infty}}{\omega^*(|h|)}$$

$$= O(1)\sup_{h\neq 0}\frac{\left(\omega(|h|)\right)^{\frac{\beta}{\eta}}}{\omega^*(|h|)}\frac{1}{\lambda^m}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Suppose $1 \leq p < \infty$. Using (4.20) the Fubini inequality [11], we get

$$\|E_{\lambda,\ m}(\cdot+h,\cdot)\|_{p} = \frac{\lambda}{(m-1)!}\left\{\int\limits_{\mathbb{R}}\left|\int\limits_{\mathbb{R}} t^m\mathcal{K}(\lambda t)\right.\right.$$

$$\left.\left.\cdot\left(\int\limits_0^1(1-u)^{m-1}\left(\Delta_{ut}f^{(m)}(x+h-t)-\Delta_{ut}f^{(m)}(x-t)\right)du\right)dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\leq \frac{\lambda}{(m-1)!}\int\limits_{\mathbb{R}}\left|t^m\mathcal{K}(\lambda t)\right|$$

$$\cdot\left\{\int\limits_{\mathbb{R}}\left|\int\limits_0^1(1-u)^{m-1}\left(\Delta_{ut}f^{(m)}(x+h-t)-\Delta_{ut}f^{(m)}(x-t)\right)du\right|^p dx\right\}^{\frac{1}{p}}dt$$

$$\leq \frac{\lambda}{(m-1)!}\int\limits_{\mathbb{R}}\left|t^m\mathcal{K}(\lambda t)\right|$$

$$\cdot\left(\int\limits_0^1(1-u)^{m-1}\left\{\int\limits_{\mathbb{R}}\left|\Delta_{ut}f^{(m)}(x+h-t)-\Delta_{ut}f^{(m)}(x-t)\right|^p dx\right\}^{\frac{1}{p}}du\right)dt.$$

It is clear that $1 \leq p < \infty$

$$\left\{\int\limits_{\mathbb{R}}\left|\Delta_{ut}f^{(m)}(x+h-t)-\Delta_{ut}f^{(m)}(x-t)\right|^p dx\right\}^{\frac{1}{p}} \leq 2\omega\left(f^{(m)};|ut|\right)_p$$

and

$$\left\{ \int\limits_{\mathbb{R}} \left| \Delta_{ut} f^{(m)} (x + h - t) - \Delta_{ut} f^{(m)} (x - t) \right|^p dx \right\}^{\frac{1}{p}} \le 2\omega \left(f^{(m)}; |h| \right)_p .$$

Using this, (1.3), (2.4) and the properties of the modulus of continuity we get that for $f \in H_m^{\omega,\, p}$

$$\|E_{\lambda,\, m} (\cdot + h, \cdot)\|_p \le \frac{\lambda}{(m-1)!} (\omega (|h|))^{\frac{\beta}{\eta}}$$

$$\cdot \int\limits_{\mathbb{R}} \left(|t|^m |\mathcal{K} (\lambda t)| \int\limits_0^1 (1-u)^{m-1} \left(\omega \left(f^{(m)}; |ut| \right)_p \right)^{1-\frac{\beta}{\eta}} du \right)$$

$$= O(1) \frac{\lambda}{(m-1)!} (\omega (|h|))^{\frac{\beta}{\eta}}$$

$$\cdot \int\limits_{\mathbb{R}} \left(|t|^m |\mathcal{K} (\lambda t)| \left(\omega \left(f^{(m)}; |t| \right)_p \right)^{1-\frac{\beta}{\eta}} \int\limits_0^1 (1-u)^{m-1} du \right) dt$$

$$= O(1) \frac{\lambda}{(m-1)!} (\omega (|h|))^{\frac{\beta}{\eta}}$$

$$\cdot \int\limits_{\mathbb{R}} \left(|t|^m |\mathcal{K} (\lambda t)| \left(\omega \left(f^{(m)}; |t| \right)_p \right)^{1-\frac{\beta}{\eta}} \int\limits_0^1 (1-u)^{m-1} du \right) dt$$

$$= (\omega (|h|))^{\frac{\beta}{\eta}} \left(\omega \left(f^{(m)}; \frac{1}{\lambda} \right)_p \right)^{1-\frac{\beta}{\eta}} \lambda \int\limits_{\mathbb{R}} \frac{|t|^m}{m!} |\mathcal{K} (\lambda t)| (1 + \lambda |t|)^{1-\frac{\beta}{\eta}} dt.$$

$$= O(1) (\omega (|h|))^{\frac{\beta}{\eta}} \left(\omega \left(\frac{1}{\lambda} \right) \right)^{1-\frac{\beta}{\eta}} \frac{1}{\lambda^m} \left(\int\limits_0^\infty u^m |\mathcal{K} (u)| du + \int\limits_0^\infty u^{m+1} |\mathcal{K} (u)| du \right)$$

$$= O(1) (\omega (|h|))^{\frac{\beta}{\eta}} \left(\omega \left(\frac{1}{\lambda} \right) \right)^{1-\frac{\beta}{\eta}} \frac{1}{\lambda^m}.$$

Similarly we can prove that

$$\|E_{\lambda,\, m} (\cdot)\|_p = O(1) \frac{1}{\lambda^m} \omega \left(f^{(m)}; \frac{1}{\lambda} \right)_p = O(1) \frac{1}{\lambda^m} \left(\omega \left(\frac{1}{\lambda} \right) \right)^{1-\frac{\beta}{\eta}}.$$

Hence for $1 \le p < \infty$

$$\|E_{\lambda,\, m} (\cdot)\|_{\omega^*,\, p} = \|E_{\lambda,\, m} (\cdot)\|_p + \sup_{h \ne 0} \frac{\|E_{\lambda,\, m} (\cdot + h, \cdot)\|_p}{\omega^* (|h|)}$$

$$= O(1) \sup_{h \ne 0} \frac{(\omega (|h|))^{\frac{\beta}{\eta}}}{\omega^* (|h|)} \left(\omega \left(\frac{1}{\lambda} \right) \right)^{1-\frac{\beta}{\eta}}.$$

Thus, the proof is completed. □

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